Newman-Penrose formalism and related topics

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1 Newman-Penrose Formulation

1.1 Advantages

- Natural framework for calculations in radiative spacetimes
- Variables have geometrical meanings
- Practical advantages in treating Petrov type-D spacetimes
- Closely related with spinor formalism

1.2 Null tetrad

Take real-valued null vectors $\mathbf{l}, \mathbf{n}$ and complex conjugate null vectors $\mathbf{m}, \overline{\mathbf{m}}$, which satisfy

$$\mathbf{l}\cdot\mathbf{n} = l^a n_a = l_a n^a = 1 \tag{1a}$$

$$\mathbf{m}\cdot\overline{\mathbf{m}} = m^a \overline{m}_a = m_a \overline{m}^a = -1 \tag{1b}$$

$$\text{and else } 0 \tag{1c}$$

This set of null basis $(l^a, n^a, m^a, \overline{m}^a)$ has relations with orthogonal tetrad basis (t^a, x^a, y^a, z^a)

$$l^a = o^A o^{A'} = \frac{1}{\sqrt{2}}(t^a + z^a), \tag{2a}$$

$$n^a = \iota^A \iota^{A'} = \frac{1}{\sqrt{2}}(t^a - z^a), \tag{2b}$$

$$m^a = o^A \iota^{A'} = \frac{1}{\sqrt{2}}(x^a - iy^a), \tag{2c}$$

$$\overline{m}^a = \iota^A o^{A'} = \frac{1}{\sqrt{2}}(x^a + iy^a), \tag{2d}$$

where I also put spinor expressions (o^A, ι^A) of those.
Metric g_{ab} will be recovered by

$$g_{ab} = \eta_{ij} e^i{}_a e^j{}_b = 2l_{(a}n_{b)} - 2m_{(a}\overline{m}_{b)} \tag{3}$$

$$= t_a t_b - x_a x_b - y_a y_b - z_a z_b \tag{4}$$

where $\eta_{ij} = \eta^{ij} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix}$.

1.3 Connection coefficients

∇ (deriv. op.)	$m^a\nabla l_a$ $\Leftrightarrow o^A\nabla o_A$	$\frac{1}{2}(n^a\nabla l_a - \overline{m}^a\nabla m_a)$ $\Leftrightarrow o^A\nabla\iota_A = \iota^A\nabla o_A$	$-\overline{m}^a\nabla n_a$ $\Leftrightarrow \iota^A\nabla\iota_A$
$D = l^a\nabla_a$ $\Leftrightarrow o^A\bar{o}^{A'}\nabla_{AA'}$	$\kappa\ \{3,1\}$	ϵ	$\pi\ (-\tau')\ \{-1,1\}$
$\Delta = n^a\nabla_a$	$\tau\ \{1,-1\}$	$\gamma\ (-\epsilon')$	$\nu\ (-\kappa')\ \{-3,-1\}$
$\delta = m^a\nabla_a$	$\sigma\ \{3,-1\}$	β	$\mu\ (-\rho')\ \{-1,-1\}$
$\delta = \bar{m}^a\nabla_a$	$\rho\ \{1,1\}$	$\alpha\ (-\beta')$	$\lambda\ (-\sigma')\ \{-3,1\}$

1.4 Weyl scalars

The Weyl curvature C_{abcd} is defined as

$$C_{abcd} = R_{abcd} - g_{a[c}R_{d]b} + g_{b[c}R_{d]a} - \frac{1}{3}Rg_{a[c}g_{d]b}, \tag{5}$$

or alternatively by using $\Lambda = R/24, \Phi_{ab} = -\frac{1}{2}R_{ab} + \frac{1}{8}Rg_{ab}$,

$$C_{abcd} = R_{abcd} + 2\Phi_{a[c}g_{d]b} - 2\Phi_{b[c}g_{d]a} + 8\Lambda g_{a[c}g_{d]b}. \tag{6}$$

The 10 components of Weyl curvature are expressed by the following 5 complex scalars;

$$\Psi_0 \equiv \psi_{ABCD}o^Ao^Bo^Co^D = C_{abcd}l^am^bl^cm^d, \tag{7a}$$
n^a-directed transverse component, $\{4,0\}$

$$\Psi_1 \equiv \psi_{ABCD}o^Ao^Bo^C\iota^D = C_{abcd}l^an^bl^cm^d, \tag{7b}$$
n^a-directed longitudinal component $\{2,0\}$

$$\Psi_2 \equiv \psi_{ABCD}o^Ao^B\iota^C\iota^D = C_{abcd}l^am^b\bar{m}^cn^d, \tag{7c}$$
'Coulomb' component, $\{0,0\}$

$$\Psi_3 \equiv \psi_{ABCD}o^A\iota^B\iota^C\iota^D = C_{abcd}l^an^b\bar{m}^cn^d, \tag{7d}$$
l^a-directed longitudinal component, $\{-2,0\}$

$$\Psi_4 \equiv \psi_{ABCD}\iota^A\iota^B\iota^C\iota^D = C_{abcd}n^a\bar{m}^bn^c\bar{m}^d, \tag{7e}$$
l^a-directed transverse component, $\{-4,0\}$

1.5 Ricci tensor components

Real valued $(\Phi_{00},\Phi_{11},\Phi_{22},\Lambda)$ and Complex valued conjugate pairs (Φ_{01},Φ_{10}), (Φ_{02},Φ_{20}) and (Φ_{12},Φ_{21}). Total 10 real components.

$$\Phi_{00} = \Phi_{ABA'B'}o^Ao^B\bar{o}^{A'}\bar{o}^{B'} = -\tfrac{1}{2}R_{ab}l^al^b, \quad \{2,2\} \tag{8a}$$
$$\Phi_{01} = \Phi_{ABA'B'}o^Ao^B\bar{o}^{A'}\bar{\iota}^{B'} = -\tfrac{1}{2}R_{ab}l^am^b, \quad \{2,0\} \tag{8b}$$
$$\Phi_{02} = \Phi_{ABA'B'}o^Ao^B\bar{\iota}^{A'}\bar{\iota}^{B'} = -\tfrac{1}{2}R_{ab}m^am^b, \quad \{2,-2\} \tag{8c}$$
$$\Phi_{10} = \Phi_{ABA'B'}o^A\iota^B\bar{o}^{A'}\bar{o}^{B'} = -\tfrac{1}{2}R_{ab}l^a\bar{m}^b, \quad \{0,2\} \tag{8d}$$
$$\Phi_{11} = \Phi_{ABA'B'}o^A\iota^B\bar{o}^{A'}\bar{\iota}^{B'} = -\tfrac{1}{2}R_{ab}l^an^b + 3\Lambda, \quad \{0,0\} \tag{8e}$$
$$\Phi_{12} = \Phi_{ABA'B'}o^A\iota^B\bar{\iota}^{A'}\bar{\iota}^{B'} = -\tfrac{1}{2}R_{ab}m^an^b, \quad \{0,-2\} \tag{8f}$$
$$\Phi_{20} = \Phi_{ABA'B'}\iota^A\iota^B\bar{o}^{A'}\bar{o}^{B'} = -\tfrac{1}{2}R_{ab}\bar{m}^a\bar{m}^b, \quad \{-2,+2\} \tag{8g}$$
$$\Phi_{21} = \Phi_{ABA'B'}\iota^A\iota^B\bar{o}^{A'}\bar{\iota}^{B'} = -\tfrac{1}{2}R_{ab}\bar{m}^an^b, \quad \{-2,0\} \tag{8h}$$
$$\Phi_{22} = \Phi_{ABA'B'}\iota^A\iota^B\bar{\iota}^{A'}\bar{\iota}^{B'} = -\tfrac{1}{2}R_{ab}n^an^b, \quad \{-2,-2\} \tag{8i}$$

1.6 Commutators acting on a scalar function

Derived from $\nabla_{[a}\nabla_{b]}\psi = 0$. Useful to reduce derivatives.

$$\begin{aligned}
(\Delta D - D\Delta)\psi &= [(\gamma+\overline{\gamma})D + (\epsilon+\overline{\epsilon})\Delta - (\pi+\overline{\tau})\delta - (\overline{\pi}+\tau)\overline{\delta}]\psi \\
(\delta D - D\delta)\psi &= [(\overline{\alpha}+\beta-\overline{\pi})D + \kappa\Delta - (\overline{\rho}+\epsilon-\overline{\epsilon})\delta - \sigma\overline{\delta}]\psi \\
(\delta\Delta - \Delta\delta)\psi &= [-\overline{\nu}D + (\tau-\overline{\alpha}-\beta)\Delta + (\mu-\gamma+\overline{\gamma})\delta - \overline{\lambda}\overline{\delta}]\psi \\
(\overline{\delta}\delta - \delta\overline{\delta})\psi &= [(\overline{\mu}-\mu)D + (\overline{\rho}-\rho)\Delta + (\alpha-\overline{\beta})\delta - (\overline{\alpha}+\beta)\overline{\delta}]\psi
\end{aligned}$$

1.7 Field equations

Expressions of Riemann tensor. 18 complex equations, but with 16 hidden eliminant relations, in total 20 degrees of freedom.

$$D\rho - \overline{\delta}\kappa = (\rho^2+\sigma\overline{\sigma}) + (\epsilon+\overline{\epsilon})\rho - \overline{\kappa}\tau - \kappa(3\alpha+\overline{\beta}-\pi) + \Phi_{00} \tag{1a}$$
$$D\sigma - \delta\kappa = \cdots \tag{1b}$$
$$D\tau - \Delta\kappa = \cdots \tag{1c}$$

$$D\alpha - \overline{\delta}\epsilon = \cdots \tag{1d}$$
$$\cdots = \cdots \tag{1e}$$

1.8 Bianchi id.

20 identities. (In vacuum, 16).

$$\begin{aligned}
&D\Psi_1 - \overline{\delta}\Psi_0 - D\Phi_{01} + \delta\Phi_{00} \\
&\quad = (\pi - 4\alpha)\Psi_0 + 2(2\rho + \epsilon)\Psi_1 - 3\kappa\Psi_2 - (\overline{\pi} - 2\overline{\alpha} - 2\beta)\Phi_{00} \\
&\qquad -2(\overline{\rho} + \epsilon)\Phi_{01} - 2\sigma\Phi_{10} + 2\kappa\Phi_{11} + \overline{\kappa}\Phi_{02}
\end{aligned} \tag{2a}$$
$$\Delta\Psi_0 - \delta\Psi_1 + D\Phi_{02} - \delta\Phi_{01} = \cdots \tag{2b}$$
$$\cdots = \cdots \tag{2c}$$

1.9 Geometrical meaning of variables

When we consider the propagation of the basis vectors along **l** or **n**, the physical significance of the spin coefficiencies becomes apparent. The propagation properties of null congruence of null geodesics are given by

$$\tfrac{1}{2}l^i{}_{;i} = -\tfrac{1}{2}(\rho + \overline{\rho}) = -\Re\rho \equiv \theta \tag{3}$$
$$\tfrac{1}{2}l_{[i;j]}l^{i;j} = -\frac{1}{4}(\rho - \overline{\rho})^2 = (\Im\rho)^2 \equiv \omega^2 \tag{4}$$
$$\tfrac{1}{2}l_{(i;j)}l^{i;j} = \theta^2 + |\sigma|^2 \tag{5}$$

θ, ω and σ are called *optical scalars.*

1.10 Counting degrees of freedom

Degrees of tetrad transformations. (6 degrees of Lorentz boosts.)

Type I $\mathbf{l} \to \mathbf{l}, \mathbf{m} \to a\mathbf{l}, \overline{\mathbf{m}} \to \overline{\mathbf{m}} + \overline{a}\mathbf{l}$, and $\mathbf{n} \to \mathbf{n} + \overline{a}\mathbf{m} + a\overline{\mathbf{m}} + a\overline{a}\mathbf{l}$.

Type II $\mathbf{n} \to \mathbf{n}, \mathbf{m} \to b\mathbf{n}, \overline{\mathbf{m}} \to \overline{\mathbf{m}} + \overline{b}\mathbf{n}$, and $\mathbf{l} \to \mathbf{l} + \overline{b}\mathbf{m} + b\overline{\mathbf{m}} + b\overline{b}\mathbf{n}$.

Type III $\mathbf{l} \to A^{-1}\mathbf{l}, \mathbf{n} \to A\mathbf{n}, \mathbf{m} \to e^{i\theta}\mathbf{m}$, and $\overline{\mathbf{m}} \to e^{-i\theta}\overline{\mathbf{m}}$.

- Counting degrees of freedom in vacuum.
  variables: Ψ 10, spin coef. 24, and boost 6.
  equation: field eq. 12×2, Bianchi id. 16.
- Counting degrees of freedom with matter.
  variables: curvature 20, spin coef. 24. equation: field eq. 20, Bianchi 20, and $\nabla_a T^a_{\ b} = 0$, 4.

2 Principal Null Directions

2.1 Weyl & dual of Weyl tensor

$$\begin{aligned}
C_{abcd} &= C_{AA'BB'CC'DD'} \\
&= \Psi_{ABCD}\bar{\epsilon}_{A'B'}\bar{\epsilon}_{C'D'} + \bar{\Psi}_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'}
\end{aligned} \tag{6}$$
$$^*C_{abcd} = \frac{1}{2}\varepsilon_{ab}{}^{mn}C_{mncd} \tag{7}$$
$$C_{abcd} + i\,^*C_{abcd} = 2\Psi_{ABCD}\bar{\epsilon}_{A'B'}\bar{\epsilon}_{C'D'} \tag{8}$$

2.2 Canonical decomposition of symmetric spinor

Spinor $\psi_{ABCD} = \psi_{(ABCD)}$ is expressed by principal null spinor $\alpha_A, \beta_A, ...$ by using spinor characters,

$$\psi_{ABCD} = \psi_{(ABCD)} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}, \tag{9}$$

and principal null vectors are defined as real vectors constructed by $\alpha_A\bar{\alpha}_{A'}, \beta_A\bar{\beta}_{A'}, ...$ PNDs are those directions.

2.3 Petrov classification

Classification of vacuum spacetime by degeneracy of PNDs.

$$0 = \Psi_{ABCD}(z\iota^A + o^A)(z\iota^B + o^B)(z\iota^C + o^C)(z\iota^D + o^D)$$
$$0 = \Psi_4 z^4 + 4\Psi_3 z^3 + 6\Psi_2 z^2 + 4\Psi_1 z + \Psi_0 \tag{10}$$

Type I $\{1,1,1,1\}$ no coincide PNDs. algebraically general.

Type II $\{2,1,1\}$ two coincide PNDs. algebraically special (all below).

Type D $\{2,2\}$ two different pairs. Schwarzschild, Kerr, ...

Type III $\{3,1\}$ three coincide PNDs.

Type N $\{4\}$ all four coincide PNDs. Plane-wave

3 Projections of Weyl curvature

Gunnarsen-Shinkai-Maeda [GSM] proposed a transformation formula of Weyl scalar Ψ_i from ADM variables (γ_{ij}, K_{ij}), motivated by an application to interpret numerically generated spacetime.

Let $(\mathcal{M}, \eta_{ab})$ be real, 4-dimensional Lorentz vector space with volume form ε_{abcd}; $\varepsilon_{abcd}\varepsilon^{abcd} = -4!$. Let (t^a, x^a, y^a, z^a) be orthonormal basis of $(\mathcal{M}, \eta_{ab})$, and define

$$t^a t_a = +s \quad (s = \pm 1), \quad \varepsilon_{abc} = \varepsilon^{abcd} t_d, \tag{11}$$

where the tensor field $\varepsilon_{abc} = \varepsilon_{[abc]}$ satisfies $\varepsilon_{abc}\varepsilon^{abc} = 3!$ We formulate our equations in the signatures both $(+,-,-,-)$ and $(-,+,+,+)$ by choosing $s = 1$ or -1, respectively[1], because the former notation is common in working with the spinors.

First, we define the Weyl curvature C_{abcd} by (5) and decompose those into its electric and a magnetic components,

$$E_{ab} \equiv - \ C_{ambn} t^m t^n, \quad B_{ab} \equiv - \ {}^*C_{ambn} t^n t^m, \tag{12}$$

where ${}^*C_{abcd} = \frac{1}{2}\varepsilon_{ab}{}^{mn} C_{mncd}$ is a dual of the Weyl tensor. These decomposed elements E_{ab} and B_{ab} are also presented by the 3-metric γ_{ab} and the extrinsic curvature K_{ab} as [2]

$$\begin{aligned} E_{ab} &= R_{ab} - K_a{}^m K_{bm} + K K_{ab} - \frac{2}{3}\Lambda\gamma_{ab}, & (13a)\\ B_{ab} &= \varepsilon_a{}^{mn} D_m K_{nb}. & (13b) \end{aligned}$$

This is why we emphasize that our inputs are '3+1' elements. It follows from two constraint equations that the fields E_{ab}, B_{ab} are both trace-free and symmetric. We can reconstruct the Weyl curvature completely from E_{ab} and B_{ab} by

$$C_{abcd} = 4t_{[a}E_{b][c}t_{d]} + 2\varepsilon_{ab}{}^m B_{m[c}t_{d]} + 2\varepsilon_{cd}{}^m B_{m[a}t_{b]} + \varepsilon_{ab}{}^m \varepsilon_{cd}{}^n E_{mn}. \tag{14}$$

The next step is to choose a unit vector field $\hat{z}^a$ on Σ, and to decompose E_{ab}, B_{ab} into components along and perpendicular to $\hat{z}^a$. We set

$$\begin{aligned} e &= E_{ab}\hat{z}^a\hat{z}^b, & (15a)\\ e_a &= E_{bc}\hat{z}^b(\delta_a{}^c + s\hat{z}_a\hat{z}^c), & (15b)\\ e_{ab} &= E_{cd}(\delta_a{}^c + s\hat{z}_a\hat{z}^c)(\delta_b{}^d + s\hat{z}_b\hat{z}^d) + \frac{1}{2}e s_{ab}, & (15c)\\ b &= B_{ab}\hat{z}^a\hat{z}^b, & (15d)\\ b_a &= B_{bc}\hat{z}^b(\delta_a{}^c + s\hat{z}_a\hat{z}^c), & (15e)\\ b_{ab} &= B_{cd}(\delta_a{}^c + s\hat{z}_a\hat{z}^c)(\delta_b{}^d + s\hat{z}_b\hat{z}^d) + \frac{1}{2}b s_{ab}, & (15f) \end{aligned}$$

where $s_{ab} = \gamma_{ab} - \hat{z}_a\hat{z}_b$. We note that E_{ab}, B_{ab} is again reconstructed from (15a)-(15f)

$$\begin{aligned} E_{ab} &= e\hat{z}_a\hat{z}_b + 2e_{(a}\hat{z}_{b)} + e_{ab} - \frac{1}{2}s_{ab}e. & (16)\\ B_{ab} &= b\hat{z}_a\hat{z}_b + 2b_{(a}\hat{z}_{b)} + b_{ab} - \frac{1}{2}s_{ab}b. & (17) \end{aligned}$$

Such decompositions will be useful to discuss the effects of curvatures on the transversal plane to the $\hat{z}^a$ direction.

We put a rotation operator on the plane spanned by $\hat{x}_a$ and $\hat{y}_a$ as,

$$J_a{}^b \equiv \varepsilon_a{}^{bcd}\hat{z}_c t_d. \tag{18}$$

It is easy to check this mapping preserves s_{ab}, and is also easy to check $J_a{}^c J_c{}^b = -(\delta_a{}^b + s\hat{z}_a\hat{z}^b)$, which shows us $J_a{}^b$ has a complex structure, i.e., $J_a{}^b$ lets us define complex multiples of vectors $x^a \in P_z$, according to the formula $(m+in)x^a = mx^a + nJ_b{}^a x^b$. In short, $J_a{}^b$ expresses a rotation by 90 degrees in the plane orthogonal to $\hat{z}^a$.

Finally, substitute (14) and (2a)-(2d) into (7a)-(7e), we get Ψ_i using (15a)-(15f) and (18):

$$\begin{aligned} \Psi_0 &= -(e_{ab} + sJ_a{}^c b_{bc})m^a m^b, & (19a)\\ \Psi_1 &= -\frac{s}{\sqrt{2}}(e_a + sJ_a{}^c b_c)m^a, & (19b)\\ \Psi_2 &= -\frac{1}{2}(e + ib), & (19c)\\ \Psi_3 &= -\frac{s}{\sqrt{2}}(e_a - sJ_a{}^c b_c)\bar{m}^a, & (19d)\\ \Psi_4 &= -(e_{ab} - sJ_a{}^c b_{bc})\bar{m}^a\bar{m}^b. & (19e) \end{aligned}$$

[1] That is, the metric is $\eta_{ab} = s(t_a t_b - x_a x_b - y_a y_b - z_a z_b)$.

[2] L. Smarr, Ann. N.Y. Acad. of Sci. **302**, 569 (1977).

4 Gerosh-Held-Penrose formalism

4.1 Advantages

- Concept of spin- and boost-weighted quantity is introduced
- New differentiation operaters $\eth$ and þ are introduced
- Reduce number of equations and variables in NP formulation
- Tools for checking equations in NP formulation
- Simplify expressions such as 2-surface.

4.2 Spin- and Boost-weighted type

Freedoms of boosts and spatial rotations which preserves two null directions,

$$l^a \to r l^a, \quad n^a \to r^{-1} n^a, \quad m^a \to e^{i\theta} m^a \tag{20}$$

and define complex number $\lambda^2 = re^{i\theta}$. Then, if a scalar obeys transformations

$$\eta \to \lambda^p \overline{\lambda}^q \eta \tag{21}$$

then call such a scalar as *spin- and boost-weighted type* $\{p, q\}$. The *spin-weight* is $\frac{1}{2}(p-q)$ and the *boost-weight* is $\frac{1}{2}(p+q)$.

Modify original NP derivative operators $D, \Delta(=D'), \delta, \overline{\delta}(=\delta')$ as

$$\begin{aligned}
\text{þ}\,\eta &\equiv (D - p\epsilon - q\overline{\epsilon})\eta \quad \{1,1\} &(22a)\\
\text{þ}'\,\eta &\equiv (D' + p\epsilon' + q\overline{\epsilon}')\eta \quad \{-1,-1\} &(22b)\\
\eth\eta &\equiv (\delta - p\beta + q\overline{\beta})\eta \quad \{1,-1\} &(22c)\\
\eth'\eta &\equiv (\delta' + p\beta' - q\overline{\beta})\eta \quad \{-1,1\} &(22d)
\end{aligned}$$

and all equations in NP are reduced using þ (thorn) and $\eth$ (edth). (These are the phonetic symbols for the soft and hard "th".)

4.3 Prime- and Astarisk- operation

- Prime-operation

$$\begin{aligned}
&(o^A)' \to i\iota^A, \ (\iota^A)' \to io^A, \ (\overline{o}^{A'})' \to -i\overline{\iota}^{A'}, \ (\overline{\iota}^{A'})' \to -i\overline{o}^{A'} &(23a)\\
&(l^a)' = n^a, \quad (n^a)' = l^a, \quad (m^a)' = \overline{m}^a, \quad \overline{m}^{a\prime} = m^a &(23b)
\end{aligned}$$

$$(\eta')' = (-1)^{p+q}\eta \tag{24}$$

- Astarisk-operation

$$\begin{aligned}
&o^A \to o^A, \quad \iota^A \to \iota^A, \quad \overline{o}^{A'} \to \overline{\iota}^{A'}, \quad \overline{\iota}^{A'} \to -\overline{o}^{A'} &(25a)\\
&(l^a)^* = m^a, \ (n^a)^* = -\overline{m}^a, \ (m^a)^* = -l^a, \ \overline{m}^{a*} = n^a &(25b)
\end{aligned}$$

$$(\eta^*)^* = (-1)^q\eta, \quad (\eta')^* = (-1)^q(\eta^*)', \quad \overline{\eta^*} = (-i)^{p+q}(\overline{\eta}')^*, \tag{26}$$

where η is type $\{p, q\}$, and η^* is of type $\{p, -q\}$.

A Spinor

A set of non-zero 2-component vector (o, ι) is called *spin basis*, if their skew-scalar product obeys $[o, \iota] = 1$, where $[,]$ is defined by $[\xi, \eta] = -[\eta, \xi] = \epsilon_{AB}\xi^A\eta^B$. This condition is also expressed by

$$\epsilon_{AB}o^Ao^B = \epsilon_{AB}\iota^A\iota^B = 0, \quad \epsilon_{AB}o^A\iota^B = o_A\iota^A = 1. \tag{27}$$

In components with $o^A = (1,0), \iota^A = (0,1)$,

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^{AB} = o^A\iota^B - \iota^A o^B, \quad \epsilon^{AB}\epsilon_{BC} = -\delta^A_C \tag{28}$$

When raise and lower indices, be careful the position of them.

$$\begin{aligned}
\xi_A &\equiv \epsilon_{BA}\xi^B = -\epsilon_{AB}\xi^B = \xi^B\epsilon_{BA} &(29a)\\
\xi^A &= \epsilon^{AB}\xi_B = -\epsilon^{BA}\xi_B = \xi_B\epsilon^{AB} &(29b)
\end{aligned}$$

If the contraction is across adjacent *northwest* and *southeast* indices, no minus signs occur.

Theorem 1 *Every non-vanishing real null vector k^a can be written in one are other of the forms*

$$k^a = \pm\kappa^A\overline{\kappa}^{A'} \tag{30}$$

A relation to orthonormal tetrad $g_{\hat{a}\hat{b}} = \mathrm{diag}(1,-1,-1,-1)$ is given by

$$\epsilon_{AB}\overline{\epsilon}_{A'B'} = g_{\hat{a}\hat{b}}\sigma^{\hat{a}}{}_{AA'}\sigma^{\hat{b}}{}_{BB'} \tag{31}$$

where $\sigma^{\hat{a}}{}_{AA'}$ are the Infeld-van der Warden symbols, and these are related to the Pauli matrix with $\sigma_{\hat{a}}{}^{AA'} = (1/\sqrt{2})\sigma_{\hat{a}} \quad (a = 0,1,2,3)$ where $\sigma_{\hat{0}} = I$ and

$$\sigma_{\hat{1}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{\hat{2}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{\hat{3}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{32}$$